Thermal Stresses in an Orthotropic, Edge-Grain-Orientated Cylinder

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The severity of present-day, high-temperature environments has resulted in a search for materials that can maintain structural integrity at high temperatures. The anisotropic materials have received much attention because of their highly directional properties. Some pyrolytically deposited (orthotropic) materials have exhibited increased strength at high temperatures, but, because of their anisotropic nature, severe thermal stresses can also develop. An understanding of these thermal stresses becomes important if these materials are to be used in structures that must perform in high-temperature media. In this paper the engineer is provided with the equations necessary to determine the thermal stresses in an orthotropic, edge-grain-oriented material that takes the shape of a long, hollow cylinder.

Nomenclature

α = coefficient of thermal expansion, in./in. °F
a,b,c = planar coordinates in Fig. 1; also, a = inside radius, b = outside radius in Fig. 4 and Eq. (10)
A,B,C = terms defined following Eq. (9)
E = modulus of elasticity, psi
G = shear modulus, psi
r = radius, in.
T = temperature increase above reference value
T_1 = highest temperature felt by entire thickness (Fig. 4)
T_i = difference between highest temperature obtained by material (at a) and T_1; $T_i = (T_2 - T_1)$ (Fig. 4)
u = displacement of cylindrical surface at radius r, in.
x,y,z = coordinates in Figs. 2 and 3
ϵ = strain, in./in.
μ = Poisson's ratio
$\sigma_x,\sigma_y,\sigma_z$ = normal stresses (Fig. 3), psi

Introduction

The extremely high-temperature environments that exist today have made it essential to find materials that will survive these environments. Of late there has been much interest in the anisotropic materials for such applications. An "anisotropic material" is commonly defined as one having properties that vary according to the direction in which they are taken. These particular materials may be formed by vapor deposition as, for example, pyrolytic graphite. Such materials also have been called transversely isotropic, transversely anisotropic, monotropic, and orthotropic. The pyrolytically deposited (orthotropic) materials are special cases of anisotropic materials. They exhibit symmetry with respect to the c axis, the axis that is perpendicular to the planes of deposition.

An understanding of the thermal stresses developed at elevated temperatures in anisotropic materials is necessary in order to produce an adequate design. Conventional equations that are used to determine the thermal stresses in isotropic materials are, of course, not adequate in the anisotropic case. The purpose of this paper is to provide the equations that will enable the engineer to determine the thermal stresses in a long, hollow cylinder made from an orthotropic material. The general anisotropic case will be considered first; then the necessary assumptions will be made which will transform the general anisotropic equations for use in the edge-grain-oriented, pyrolytically deposited materials.

Solutions to the general three-dimensional equations of thermoelasticity must satisfy the equations of equilibrium and compatibility. The differences between the isotropic and anisotropic equations of thermoelasticity lie in the stress-strain relationships.

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Analysis

General Anisotropic Case

Consider a three-dimensional element subject to three normal stresses: σ_x, σ_y, and σ_z. If a temperature increase is also felt by the element, the stress-strain relationships are as follows:

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\mu_{yx}\sigma_y}{E_y} - \frac{\mu_{zx}\sigma_z}{E_z} + \alpha_x T \\
\epsilon_y &= \frac{\sigma_y}{E_y} - \frac{\mu_{xy}\sigma_x}{E_x} - \frac{\mu_{zy}\sigma_z}{E_z} + \alpha_y T \\
\epsilon_z &= \frac{\sigma_z}{E_z} - \frac{\mu_{xz}\sigma_x}{E_x} - \frac{\mu_{yz}\sigma_y}{E_y} + \alpha_z T \\
\epsilon_{xy} &= \frac{\sigma_{xy}}{G_{xy}} \qquad \epsilon_{yz} = \frac{\sigma_{yz}}{G_{yz}} \qquad \epsilon_{zx} = \frac{\sigma_{zx}}{G_{zx}}
\end{aligned} \tag{1}$$

where the general term $\mu_{ab}(\sigma_a/E_a)$ signifies the strain in the b direction when loaded in tension by σ_a in the a direction.

Orthotropic Case

Figure 1 shows the way the layers of a vapor-deposited pyrolytic material are oriented. The direction parallel to the planes of deposition is referred to as the ab direction, and the direction perpendicular to the planes is designated the c direction. If plane ab is the plane of deposition and the properties in the a direction are assumed to be the same as those in the b direction, one may say the material possesses elastic symmetry with respect to the c axis. Thus $E_a = E_b$, $\mu_{ca} = \mu_{cb}$, $\mu_{bc} = \mu_{ac}$, $\mu_{ab} = \mu_{ba}$, and $\alpha_b = \alpha_a$. Applying these assumptions to the stress-strain relationship, Eqs. (1), and noting $x = a$, $y = b$, and $z = c$,

$$\epsilon_x = \frac{1}{E_x}\left(\sigma_x - \mu_{xy}\sigma_y - \frac{\mu_{zx}E_x\sigma_z}{E_z}\right) + \alpha_x T \tag{2a}$$

$$\epsilon_y = \frac{1}{E_x}\left(\sigma_y - \mu_{xy}\sigma_x - \frac{\mu_{zx}E_x\sigma_z}{E_z}\right) + \alpha_x T \tag{2b}$$

$$\epsilon_z = \sigma_z/E_z - \mu_{xz}/E_x(\sigma_x + \sigma_y) + \alpha_z T \tag{2c}$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \qquad \epsilon_{yz} = \frac{\sigma_{yz}}{G_{yz}} \qquad \epsilon_{zx} = \frac{\sigma_{zx}}{G_{zx}} \tag{2d}$$

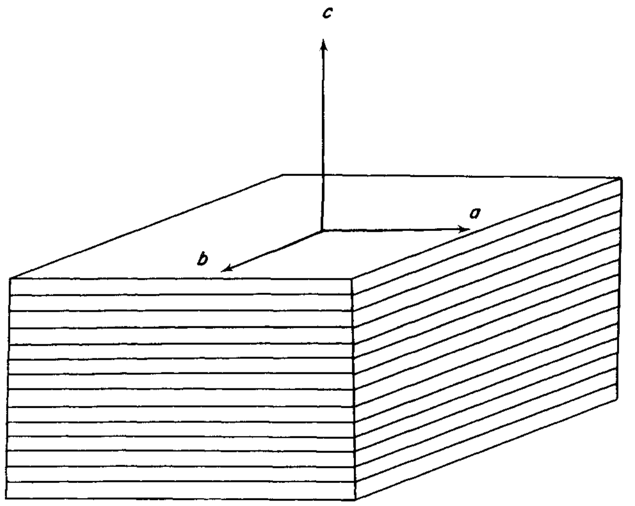

Fig. 1 Oriented layers of vapor-deposited pyrolytic material.

Figure 2 shows the geometry of the long, hollow cylinder under consideration. The cross section is circular and is subjected to a temperature distribution that is a function of radius only. It will be assumed that the temperature remains constant along the axis of the cylinder and that the inside radius is at a higher temperature than the outside.

Figure 3 shows an element of the cylinder formed by a ring cut by two planes perpendicular to the axis at a unit distance apart. The element is bounded by two axial planes and two concentric cylindrical surfaces. Because of the radial symmetry of the geometry and temperature distribution about the axis of the cylinder, it will be assumed that there are no shearing stresses. This assumption is correct if one considers that the deformation of the cross section is to remain plane if taken sufficiently distant from the ends of the cylinder. Therefore, the unit elongation ϵ_z in the direction of the axis of the cylinder is constant. These same geometrical considerations and assumptions are found in Refs. 1 and 2.

Transformation to Edge-Grain Orientation

The orthotropic nature of certain materials may be used in either of two ways. By aligning the planes of maximum thermal conduction parallel to the heat flow, a heat-sink type of structure can be made. By placing the planes of maximum conduction perpendicular to the direction of heat flow, the material acts as an insulator. In the following discussion we will examine the thermal stresses developed in a structure that utilizes the heat-sink property of the material, sometimes called edge-grain oriented.

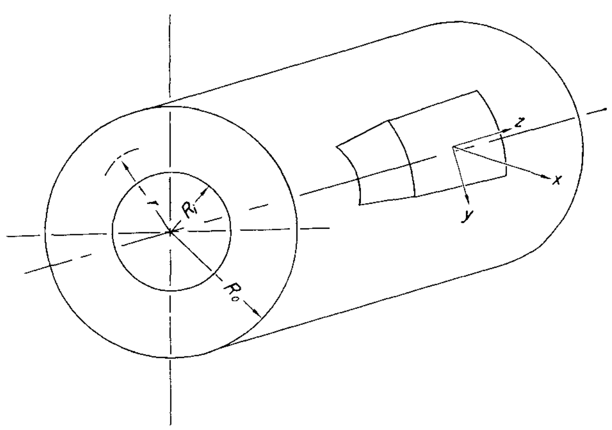

Fig. 2 Geometry of long, hollow cylinder under consideration.

In Figure 3, σ_x and σ_y denote the normal radial stress on side $m_1n_1o_1p_1$ and the tangential stress on side nn_1oo_1, respectively. σ_z is the axial stress on side mm_1nn_1. These three stresses, σ_x, σ_y, and σ_z, are functions of the radius. The corresponding equation of equilibrium for the element is[2]

$$\sigma_y - \sigma_x - r(d\sigma_x/dr) = 0 \tag{3}$$

If u denotes the displacement of the cylindrical surface $mnop$ at radius r, then the displacement of the surface $m_1n_1o_1p_1$ at radius $r + dr$ is $u + (du/dr)dr$. The net elongation in the radial direction is $(du/dr)dr$ and therefore the strain in the x direction becomes

$$\epsilon_x = du/dr \tag{4}$$

The tangential strain is equal to the unit elongation of the corresponding radius

$$\epsilon_y = ud\theta/rd\theta = u/r \tag{5}$$

The strain in the z direction is equal to a constant

$$\epsilon_z = \text{constant} \tag{6}$$

Combining Eqs. (4) and (5) yields

$$\epsilon_x = \epsilon_y + r(d\epsilon_y/dr) \tag{7}$$

Solving for σ_z in Eq. (2c) and substituting this expression into Eqs. (2a) and (2b) yields the following expressions for ϵ_x and ϵ_y:

$$\epsilon_x = (1 - \mu_{zx}\mu_{xz})\frac{\sigma_x}{E_x} - (\mu_{xy} + \mu_{zx}\mu_{xz})\frac{\sigma_y}{E_x} + T(\alpha_x + \mu_{zx}\alpha_z) - \mu_{zx}\epsilon_z \tag{8}$$

$$\epsilon_y = (1 - \mu_{zx}\mu_{xz})\frac{\sigma_y}{E_x} - (\mu_{xy} + \mu_{zx}\mu_{xz})\frac{\sigma_x}{E_x} + T(\alpha_x + \mu_{zx}\alpha_z) - \mu_{zx}\epsilon_z \tag{9}$$

letting $A = (1 - \mu_{zx}\mu_{xz})$, $B = (\mu_{xy} + \mu_{zx}\mu_{xz})$, and $C = (\alpha_x + \mu_{zx}\alpha_z)$. Substituting Eqs. (8) and (9) into (7),

$$\frac{d\epsilon_y}{dr} = \frac{A}{E_x}\frac{d\sigma_y}{dr} + A\sigma_y\frac{d(1/E_x)}{dr} - \frac{B}{E_x}\frac{d\sigma_x}{dr} - B\sigma_x\frac{d(1/E_x)}{dr} + C\frac{dT}{dr} + T\frac{dC}{dr}$$

and

$$A\frac{\sigma_y}{E_x} - B\frac{\sigma_x}{E_x} + CT - \mu_{zx}\epsilon_z + \frac{A}{E_x}r\frac{d\sigma_y}{dr} + Ar\sigma_y\frac{d(1/E_x)}{dr} - \frac{Br}{E_x}\frac{d\sigma_x}{dr} - Br\sigma_x\frac{d(1/E_x)}{dr} + Cr\frac{dT}{dr} + Tr\frac{dC}{dr} = A\frac{\sigma_x}{E_x} - B\frac{\sigma_y}{E_x} + CT - \mu_{zx}\epsilon_z$$

This is the complete differential equation involving the radial stress σ_x, the tangential stress σ_y, and the mechanical properties of the material. If the variation of mechanical properties with radius are neglected and Eq. (3) is used to replace σ_y, the resulting differential equation becomes

$$\frac{A}{E_x}r^2\frac{d^2\sigma_x}{dr^2} + \frac{3A}{E_x}r\frac{d\sigma_x}{dr} + Cr\frac{dT}{dr} = 0 \tag{10}$$

It has been assumed up to this point that the cylinder is totally unrestrained. Consequently the thermal stresses that are produced are due to the nonuniform heating of the wall. If steady-state conditions exist, the temperature distribution through the wall becomes logarithmic. A typical steady-state temperature distribution is shown in Fig. 4. It is noted that the distribution is composed of a uniform temperature T_1 and the logarithmic portion $T_i(\ln b/r)/(\ln b/a)$, the total being

$$T = T_1 + T_i(\ln b/r)/(\ln b/a) \tag{11}$$

where a and b are the inside and outside radii, and $T_i = (T_2 - T_1)$. In an isotropic material the uniform temperature T_1 does not produce thermal stress because each element of the cylinder elongates uniformly. But in an anisotropic material the situation changes, the difference being the nonuniform elongation caused by unequal thermal expansion coefficients. Differentiating Eq. (11),

$$dT/dr = -T_i/r \ln(b/a) \tag{12}$$

Substituting this expression into Eq. (10),

$$r^2(d^2\sigma_x/dr^2) + 3r(d\sigma_x/dr) = K \tag{13}$$

where

$$K = \frac{T_i}{\ln b/a}\left[\frac{E_x(\alpha_x + \mu_{zx}\alpha_z)}{(1 - \mu_{zx}\mu_{xz})}\right]$$

Solving this differential equation for σ_x yields

$$\sigma_x = P + Qr^{-2} + (K/2)\ln r \tag{14}$$

If the surfaces are assumed to be free from external forces, then the constants P and Q can be solved by considering these boundary conditions: $(\sigma_x)_{r=a} = 0$; $(\sigma_x)_{r=b} = 0$. This yields

$$Q = \frac{K}{2}\left(\frac{\ln b/a}{a^{-2} - b^{-2}}\right)$$

$$P = \frac{K}{2}\left(\frac{a^{-2}\ln b - b^{-2}\ln a}{b^{-2} - a^{-2}}\right)$$

Substituting these values into Eq. (14) gives

$$\sigma_x = \frac{K}{2}\left[\left(\frac{a^{-2}\ln b - b^{-2}\ln a}{b^{-2} - a^{-2}}\right) + \frac{1}{r^2}\left(\frac{\ln b/a}{a^{-2} - b^{-2}}\right) + \ln r\right] \tag{15}$$

Eq. (15) is the radial stress in an orthotropic, edge-grain-oriented cylinder that is heated on the inside. This stress is a function of the radius, mechanical properties, and geometry. Differentiating Eq. (15) and substituting into Eq. (3), we find

$$\sigma_y = \frac{K}{2}\left[\left(\frac{a^{-2}\ln b - b^{-2}\ln a}{b^{-2} - a^{-2}}\right) - \frac{1}{r^2}\left(\frac{\ln b/a}{a^{-2} - b^{-2}}\right) + \ln r + 1\right] \tag{16}$$

This is the tangential stress as a function of radius, derived from the assumptions listed previously in this paper.

It can be seen from Eqs. (15) and (16) that for an isotropic material (i.e., $\alpha_x = \alpha_z$, $\mu_{zx} = \mu_{xz}$) the expressions become equal to those obtained by Ref. 2.

For the orientation selected (i.e., the direction of maximum conduction is along the x axis, and the direction of maximum elongation is along the z axis), the calculation of the strain ϵ_z may have to be performed in a particular design. If we assume that the cylinder can expand freely, the constant strain ϵ_z can be found from the condition that the sum of the normal forces over the cross section of the cylinder, perpendicular to the z axis, is equal to zero. Summing these normal forces, as described in Fig. 3, we have

$$\sigma_z\{dr[r + (dr/2)]d\theta\} = 0$$

Summing the forces over the total cross section yields

$$\int_a^b \sigma_z r dr = 0 \tag{17}$$

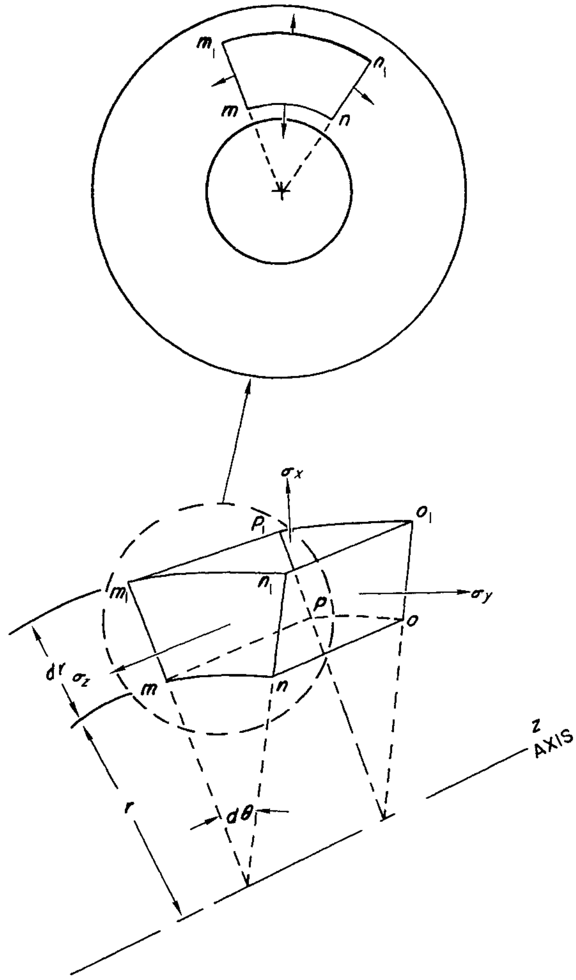

Fig. 3 Element of cylinder.

Solving for σ_z in Eq. (2c) and substituting Eqs. (15) and (16) into this expression gives

$$\sigma_z = \frac{\mu_{xz}E_z}{E_x}\left(\frac{K}{2}\right)\left[\frac{a^{-2}\ln b - b^{-2}\ln a - (1/r)^2\ln b/a}{b^{-2} - a^{-2}} + 2\ln r + 1 + \frac{a^{-2}\ln b - b^{-2}\ln a + (1/r)^2\ln b/a}{b^{-2} - a^{-2}}\right] + E_z\epsilon_z - \alpha_z E_z T \tag{18}$$

Multiplying Eq. (18) by rdr, substituting Eq. (11) into it, and integrating gives

$$\int_a^b \sigma_z r dr = 0 = \frac{\mu_{xz}E_z K}{E_x}\left(\frac{a^{-2}\ln b - b^{-2}\ln a}{b^{-2} - a^{-2}}\right)\left(\frac{b^2 - a^2}{2}\right) + E_z\epsilon_z\left(\frac{b^2 - a^2}{2}\right) + \frac{\mu_{xz}E_z K}{2E_x}\left(\frac{b^2 - a^2}{2}\right) - \frac{\alpha_z E_z T_i \ln b}{\ln b/a}\left(\frac{b^2 - a^2}{2}\right) - \alpha_z E_z T_1\left(\frac{b^2 - a^2}{2}\right) + \left(\frac{\alpha_z E_z T_i}{\ln b/a} + \frac{\mu_{xz}E_z K}{E_x}\right)\left[\frac{b^2\ln b}{2} - \frac{a^2\ln a}{2} - \left(\frac{b^2 - a^2}{4}\right)\right] \tag{19}$$

Simplifying Eq. (19):

$$0 = \frac{\mu_{xz}K}{E_x}\left(\frac{b^2 - a^2}{2}\right)\left(\frac{a^{-2}\ln b - b^{-2}\ln a}{b^{-2} - a^{-2}}\right) + \frac{\mu_{xz}K}{2E_x}(b^2\ln b - a^2\ln a) + \frac{\alpha_z T_i}{\ln b/a}\left[\frac{a^2}{2}\ln b/a - \left(\frac{b^2 - a^2}{4}\right)\right] + \epsilon_z\left(\frac{b^2 - a^2}{2}\right) - \alpha_z T_1\left(\frac{b^2 - a^2}{2}\right)$$

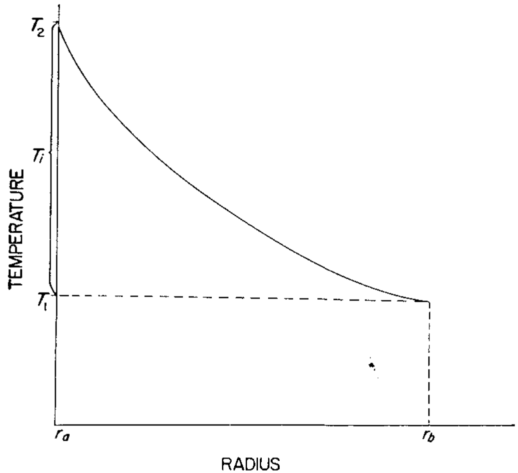

Fig. 4 Typical steady-state temperature distribution.

Solving for ϵ_z:

$$\epsilon_z = \frac{2\alpha_z T_i}{(b^2 - a^2)\ln b/a}\left(\frac{b^2 - a^2}{4} + \frac{a^2 \ln a/b}{2}\right) + \alpha_z T_1 \qquad (20)$$

Eq. (20) is the total strain in the z direction due to the temperature distribution described in Eq. (11).

Planes of Deposition Perpendicular to Heat Flow

In Ref. 1 the differential equation of thermal stress for the case where the material is used as an insulator has been obtained. This equation is

$$r^2 \frac{d^2\sigma_x}{dr^2} + 3r \frac{d\sigma_x}{dr} \psi + {}_c\sigma_x = \phi_c T - \Omega r \frac{dT}{dr} \qquad (21)$$

where

$$\psi_c = 1 - [(E_x/E_z) - (\mu_{xz})^2]/[1 - (\mu_{xz})^2]$$

$$\phi_c = \left[\left(\frac{\alpha_z}{\alpha_x} - 1\right) + \mu_{xy}\left(\frac{\mu_{xz}}{\mu_{xy}} - 1\right)\right]\frac{\Omega}{1 + \mu_{xy}}$$

$$\Omega = E_x\alpha_x/(1 - \mu_{xy})$$

This equation was developed for a long, hollow cylinder, long enough to neglect the strain in the z direction.

It may be remembered that the uniform temperature portion of the steady-state total temperature distribution was included in developing the equations of thermal stress for the edge-grain orientation. It was shown, however, that the uniform temperature did not produce any stress when the planes of deposition were aligned parallel to the direction of heat flow. It is interesting to note that the first term on the right side of Eq. (21) shows that a thermal stress is developed because of a uniform temperature environment when the planes of deposition are perpendicular to the heat flow.

Summary

It has been shown that the thermal stresses developed in orthotropic materials differ according to how the planes of deposition are aligned relative to the direction of heat flow. The classical equations developed for determining the thermal stresses in the isotropic case are not adequate when an orthotropic material is used. Thermal stresses can develop in an anisotropic material resulting from a uniform temperature change, whereas no stresses will subsequently be developed in an isotropic material. It must be remembered that the analysis presented in this paper neither assumes exterior constraint of the cylinder nor accounts for variations in the mechanical properties with temperature.

References

[1] Garber, A. M., "Pyrolytic materials for thermal protection systems," Aerospace Eng. **22,** 126–137 (1963).

[2] Timoshenko, S., *Strength of Materials* (D. Van Nostrand Co., Inc., New York, 1955), 3rd ed., Pt. 2, pp. 228–234.